Techniques of State Space Modeling

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Introduction

Over the past several decades many techniques have been used to model the PWM switch. These include both analytical and circuit based models that become unwieldy for fourth, and some second order systems. The simplest approach is to use the state space analytical method. This method may be used in conjunction with computational software, such as Matlab or Maple, to quickly and easily model a given power stage. In this paper state space modeling is presented in a step-by-step manner such that one may easily implement the approach in software by following a prescribed recipe.

State Space Modeling

State space modeling is a technique that describes a given system using a system of linear differential equations. These equations are easily manipulated using matrix operations and may be used to relate the internal, or state variables to the system input and output.

The state equations may be expressed in matrix form as the following:

$$\dot{x} = Ax + Bu$$
$$y = Qx + Ru$$

Where $\dot{x}$ is the time derivative of the state variable vector, A is the state matrix, x is the state variable vector, B is a vector, u is the input, y is the output, Q is a transposed vector relating the state variables to the output, and R is a vector relating the input to the output.

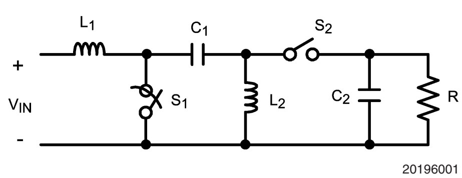

FIGURE 1. SEPIC Technology

Because a given network has two states in CCM , S1 on, S2 off and S1 off, S2 on, the response of the network in each state may be time weighted and averaged. For example, the SEPIC topology shown in Figure 1 may be redrawn for each of its two states using convenient shorthand for the internal variables as shown in Figure 2.

The equations for both states may be time weighted and averaged using the following relationships:

$$\dot{x} = \left[A_1 \frac{t_{ON}}{t_P} + A_2 \frac{t_{OFF}}{t_P}\right] x + \left[B_1 \frac{t_{ON}}{t_P} + B_2 \frac{t_{OFF}}{t_P}\right] u$$

$$y = \left[Q_1 \frac{t_{ON}}{t_P} + Q_2 \frac{t_{OFF}}{t_P}\right] x + \left[R_1 \frac{t_{ON}}{t_P} + R_2 \frac{t_{OFF}}{t_P}\right] u$$

Where the matrix subscript refers to the state of the network. Alternatively, these relationships may be expressed as the following:

$$\dot{x} = [A_1\, d + A_2\, d'] \, x + [B_1\, d + B_2 \, x\, d'] \, u$$

$$y = [Q_1 d + Q_2 d'] \, x + [R_1 d + R_2 d'] \, u$$

Where

$$d = \frac{t_{ON}}{t_p}$$

$$d' = 1 - d = \frac{t_{OFF}}{t_p}$$

The variables x, d, u, and y have both large and small signal components. Each variable in relation to its components may be expressed as:

$$x = X + \chi$$
$$d = D + \delta$$
$$u = V_{IN} + v_{in}$$
$$y = V_{OUT} + v_{out}$$

Where the first and second terms on the right hand side of the equality correspond to the large and small signal components of a given variable.

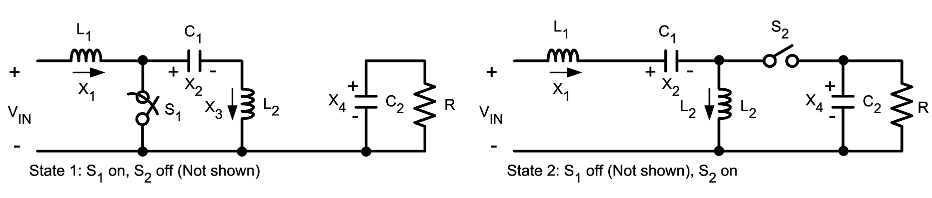

FIGURE 2. Network states 't_{on}' and 't_{off}'

State Space Modeling (Continued)

Substituting these expressions into the time weighted average expression results in the following:

$$\dot{x} = [A_1 (D+\delta) + A_2 (1-D-\delta)](X + \chi) + [B_1 (D+\delta) + B_2 (1-D-\delta)](V_{IN} + v_{in})$$

$$V_{OUT} + v_{out} = [Q_1 (D+\delta) + Q_2 (1-D-\delta)](X + x\chi) + [R_1 (D+\delta) + R_2 (1-D-\delta)](V_{IN} + v_{in})$$

Because χ, δ, v_{in}, and v_{out} are small signals, the product of two small signals will have a negligible effect on the system response. These second order terms may be ignored without impairing the result. It is also important to note that, depending on the variables of interest, assumptions may be made that change each expression. This is described in the following sections.

Large Signal Relationship

In order to determine the large signal input-to-output relationship the small signal perturbations are considered to be negligible and set equal to zero. Doing so, the large signal response may be expressed as:

$$0 = [A_1 D + A_2 D1']X + [B_1 D + B_2 D'] V_{IN}$$

$$V_{OUT} = [Q_1 D + Q_2 D']X$$

$$\frac{V_{OUT}}{V_{IN}} = -\left[Q_1 \; D + Q_2 \; D'\right]\left[A_1 \; D + A_2 \; D'\right]^{-1}\left[B_1 \; D + B_2 \; D'\right]$$

Small Signal Relationship

Determining the small signal control-to-output transfer function is analogous to the large signal case. This is to say that perturbations on the input (vin) are ignored. Doing so, the relationships used to determine the small signal control-to-output transfer function may now be expressed as:

$$\dot{\chi} = [A_1 D + A_2 D']X + [B_1 D + B_2 D'] V_{IN} + [(A_1 - A_2) + (B_1 - B_2) V_{IN}] \delta$$

$$V_{OUT} + v_{out} = [Q_1 D + Q_2 \times D']X + [Q_1 D + Q_2 \times D'] \chi + [(Q_1 - Q_2)X] \delta + [R_1(D + \delta) + R_2(1-D-\delta)]V_{IN}$$

To solve for the small signal control-to-output transfer function, the equations above must be converted to the frequency domain using the Laplace transform. The corresponding small signal control-to-output response may be expressed as:

$$\chi(s) = [s\,I - (A_1 \times D + A_2 D')]^{-1} [(A_1 - A_2)X + (B_1 - B_2) VIN] \times\delta(s)$$

The small signal control-to-output transfer function may be expressed as:

$$\frac{V_{OUT}}{\delta}(s) = \left[Q_1 \; D + Q_2 \; D'\right]\left[s \; I - (A_1 \; D + A_2 \; D')\right]^{-1} \left[(A_1 - A_2)X + (B_1 - B_2) \; V_{IN}\right] + (Q_1 - Q_2)X$$

Determining the small signal line-to-output transfer function is analogous to the small signal control-to-output case. This is to say that perturbations on the duty cycle (δ) are ignored. Doing so, the small signal line-to-output transfer function may be expressed as:

$$\frac{V_{OUT}}{V_{IN}}(s) = \left[Q_1 \; D + Q_2 \; D'\right]\left[s \; I - (A_1 \; D + A_2 \; D')\right]^{-1} B$$

Application to the SEPIC Power Stage

State equations for the SEPIC power stage, shown in Figure 2, in State 1 may be expressed as the following:

$$\dot{x}_1 = \frac{V_{IN}}{L_1}$$

$$\dot{x}_2 = \frac{x_3}{C_1}$$

$$\dot{x}_3 = \frac{x_2}{L_2}$$

$$V_{OUT} = x_4$$

In matrix form this is expressed as the following:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{C_1} & 0 \\ 0 & -\frac{1}{L_2} & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{R\,C_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} \frac{1}{L_1} \\ 0 \\ 0 \\ 0 \end{bmatrix} V_{IN}$$

$$V_{OUT} = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

By induction it becomes apparent that the matrices A_1, B_1, Q_1, and R_1 may be expressed as:

Application to the SEPIC Power Stage (Continued)

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{C_1} & 0 \\ 0 & -\frac{1}{L_2} & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{RC_2} \end{bmatrix}$$

$$B_1 = \begin{bmatrix} \frac{1}{L_1} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$Q_1 = [0\ 0\ 0\ 1]$$
$$R_1 = [0\ 0\ 0\ 0]$$

The state equations of the network, shown in Figure 2, in State 2 may be expressed as the following:

$$\dot{x}_1 = -\frac{x_2}{L_1} - \frac{x_4}{L_1} + \frac{V_{IN}}{L_1}$$

$$\dot{x}_2 = \frac{x_1}{C_1}$$

$$\dot{x}_3 = \frac{x_4}{L_2}$$

$$\dot{x}_4 = \frac{x_1}{C_2} - \frac{x_3}{C_2} - \frac{x_4}{R\ C_2}$$

$$V_{OUT} = x_4$$

In matrix form this is expressed as the following:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{L_1} & 0 & -\frac{1}{L_1} \\ \frac{1}{C_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{L_2} \\ \frac{1}{C_2} & 0 & -\frac{1}{C_2} & -\frac{1}{R\ C_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} \frac{1}{L_1} \\ 0 \\ 0 \\ 0 \end{bmatrix} V_{IN}$$

$$V_{OUT} = [\,0\ \ 0\ \ 0\ \ 1\,] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

By induction it becomes apparent that the matrices A_2, B_2, Q_2, and R_2 may be expressed as:

$$A_2 = \begin{bmatrix} 0 & -\frac{1}{L_1} & 0 & -\frac{1}{L_1} \\ \frac{1}{C_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{L_2} \\ \frac{1}{C_2} & 0 & -\frac{1}{C_2} & -\frac{1}{RC_2} \end{bmatrix}$$

$$B_2 = \begin{bmatrix} \frac{1}{L_1} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$Q_2 = [0\ 0\ 0\ 1]$$
$$R_2 = [0\ 0\ 0\ 0]$$

The large signal input-to-output relationship and associated large signal state relationships may be expressed as:

Application to the SEPIC Power Stage (Continued)

$$\frac{V_{OUT}}{V_{IN}} = \frac{D}{D'}$$

$$x_1 = \frac{D^2 \quad V_{IN}}{D'^2 \quad R}$$

$$x_2 = V_{IN}$$

$$x_3 = \frac{D \ V_{IN}}{-D' \ R}$$

$$x_4 = \frac{D \ V_{IN}}{D'}$$

The small signal control-to-output relationship may be expressed as:

$$\frac{V_{OUT}}{\delta}(s) = \frac{A_1 s^3 + A_2 s^2 + A_3 s + A_4}{A_5 s^4 + A_6 s^3 + A_7 s^2 + A_8 s + A_9} V_{IN}$$

$$A_1 = -L_1C_1L_2D$$

$$A_2 = L_1C_1RD'^2$$

$$A_3 = -D^2L_1$$

$$A_4 = D'^2R$$

$$A_5 = D'^2L_1C_1L_2C_2R$$

$$A_6 = D'^2L_1C_1L_2$$

$$A_7 = D'^2R(L_1C_1D'^2 + L_2C_2D'^2 + C_1L_2D'^2 + L_1C_2D^2$$

$$A_8 = D'^2 (L_2D'^2 + L_1D^2)$$

$$A_9 = D'^4R$$

The small signal line-to-output transfer function may be expressed as:

$$\frac{V_{OUT}}{V_{IN}}(s) = \frac{A_1 s^2 + A_2}{A_3 s^4 + A_4 s^3 + A_5 s^2 + A_6 s + A_7}$$

$$A_1 = C_1L_2RD'$$

$$A_2 = RDD'$$

$$A_3 = L_1C_1L_2C_2R$$

$$A_4 = L_1C_1L_2$$

$$A_5 = R\ (L_1C_1D'^2 + L_2C_2D'^2 + L_1C_2D^2 + C_1L_2D'^2)$$

$$A_6 = L_1D^2 + L_2D'^2$$

$$A_7 = RD'^2$$

While the small signal expressions above provide little insight into the contribution of each component to the response of the power stage, this is of little or no consequence. Components in the power stage are typically selected according to large signal, DC, requirements of the system. The small signal, AC, response is simply an artifact of component selection.

Using the Model

To take full advantage of this model, one must find the poles and zeros using numerical techniques. The transfer function poles and zeros are easily calculated using routines in computational software. Once the pole and zero locations of the power stage are found, the switcher can be compensated in the usual manner.

Conclusion

It should become apparent that the state space method of deriving equations for any given variable in terms of another variable, large signal or small signal, becomes manageable using elementary matrix operations. The necessary matrix and numerical operations are facilitated by the use of computational software. Overall this process allows one to quickly and easily derive the transfer functions and determine the response of a given network.

Notes

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